# Universality of Local Bulk Regime for Hermitian Matrix Models

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# **1** Introduction

## 1.1 Asymptotic "Philosophy" of RMT

Let  $M_n$  be a n S a 2G - 93 10. 9091 c T @ F 401 - 437.1 @ 6 9091 <math>c b

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 $nN( ) ' 1$ : local (microscopic) regime;

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# Assume Supp *N*, then

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Probability theory analogs: LLN, CLT, "collective theorems", Yu. Linnik

"emerging universality" QFT

#### 1.2 Linear Eigenvalue Statistics.

Take ': R ! R and write the *linear eigenvalue statistic*  $N_n['] := \bigvee_{l=1}^{n} ' \binom{n}{l} = \operatorname{Tr}' (M_n) = \bigcup_{R}^{Z} ' \binom{n}{l} = \operatorname{Tr}' (M_n) = \operatorname{Tr}' (M_n) = \operatorname{Tr}' \binom{n}{l} = \operatorname{Tr}' = \operatorname{Tr}' \binom{n}{l} = \operatorname{Tr}' = \operatorname{T$ 

is known as the test function. In particular

$$N_n() := ]f_{l}^{(n)} 2 ; l = 1; ...; ng$$
  
=  $X^n$   $\binom{(n)}{l} = N_n[]$ 

is the Eigenvalue Counting Measure of eigenvalues and  $N_n = n^{-1} N_n$ .

#### Define

bulk N = f 2 supp N : 9 > 0;  $\lim_{n! \to j} \sup_{j \in J} j(j) = 0g$ :

We have for  $N_{\Omega}[']$ :

' is *n*-independent: global regime;

 $'_n = '(( _0)L_n); L_n ! 1; nL_n ! 0: intermediate bulk regime;$ 

 $'_n = (( _0)n_n( _0))$ : local bulk regime

## **1.3 Typical Problems**

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(i)  $\lim_{n!} n^{-1} N_{n}$ , global regime, selfaveraging, "LLN"

(ii)  $Var f N_n g$ , global and intermediatee regime, fluctuations, "CLT"

(iii) 
$$\mathsf{P}fN_n(n) = kg; k 2 N; E_n(n) = \mathsf{P}fN_n(n) = 0g,$$

gap probability, local regime, spacings, universality

in particular  $_{n} = ( _{0}; _{0} + S = n _{n} ( _{0})); _{0} 2$  bulk *N* for the local bulk regime

#### 1.4 Hermitian Matrix Models

*n n* hermitian random matrices with the law

j=1 1 j < k n  $V : \mathbb{R}$  /  $\mathbb{R}_+$  is a continuous function (potential), and

$$9'' > 0; L < 1 V() (2 + ') \log(1 + j j) > 0; j j L$$

 $V = {}^{2}$  = 2 corresponds to the Gaussian Unitary Ensemble (GUE).

#### 1.5 Results (a collection)

(i) For any probability measure m on  $\mathbb{R}$ ;  $m(\mathbb{R}) = 1$  define (*Gauss*) Z Z Z E[m] = V()m(d)  $\log j$  jm(d)m(d);

and let N

In particular

$$V:p: \sum_{\text{supp } N} \frac{()d}{()} = V^{\theta}()=2; 2 \text{ supp } N:$$

i.e., an analog of the LLN : *Wigner, 52; Brezin et al, 79; A. Boutet de Monvel, P., Shcherbina, 95; Deift et al 98; Johansson, 98; P., Shcherbina, 07.* 

(ii)  $\operatorname{Var} f N_n['] g$  does not grow with *n* if ' 2



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i.e., an analog of the LLN : Wigner, 52; Brezin et al, 79; A. Boutet de Monve, P., Shcherbina, 95; Deift et al 98; Johansson, 98; P., Shcherbina, 07.te.  $V^{\theta}$  is Lip 1; there exists a closed interval  $[a; b] = \operatorname{supp} N$  such that  $\sup_{2[a;b]} jV^{\theta \theta}(\ )j \quad C_1 < 1; \quad 0 < \inf_{2[a;b]} (\ ):$ 

Then we have for any d > 0:

*(i)* 

$$\sup_{2[a+d;b \ d]} j_{n}() \quad ()j \ Cn^{2=9};$$

i.e.,  $[a + d; b \quad d]$  belongs to bulk N;

(ii) if  $p_l^{(n)}$ ; l = 1; 2; ... are the marginals of the joint probability density of eigenvalues, then for any  $_0 2 [a + d; b \quad d]$ 

$$\lim_{n! \to I} [n(0)] p_{I}^{(n)} + \frac{X_{1}}{n(0)} = 0 + \frac{X_{I}}{n(0)}$$

(iii) if  $E_n(n) = \mathbf{P} f_l^{(n)} \mathbf{2}_{n'}$ ,  $l = 1, \dots, ng$  is the gap probability of the ensemble and  $n = (0, 0 + S = n(0))_0 \mathbf{2} [a + d, b d],$ then

$$\lim_{n! \to 1} \mathbf{P}(n) = \det(1 \quad S(s));$$

where

$$(S(s)f)(x) = \int_{0}^{Z} \frac{\sin(x - y)}{(x - y)} f(y) dy; \ x \ 2 \ [0; s]:$$

Dyson, 61, 73; P., Shcherbina, 97, 07; Deift et al 99

More if  $\begin{pmatrix} 0 \end{pmatrix} = 0$ , 7 (singular points (e.g. edge) universality).

# 2 Proof (outline)

### 2.1 Orthogonal Polynomials Techniques

Weyl integration formula for the joint eigenvalue density

$$p_n(1; ...; n) = Q_n^1 e^{n P_{k=1}^n V(1)}$$

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Weyl integration formula for the joint eigenvalue density

*p*<sub>n</sub>

$$\begin{array}{c} (n) \\ I \\ \end{array} = e^{-nV=2} P_{I}^{(n)} \text{ and } K_{n}(;) = \begin{array}{c} P_{I-1} & (n) \\ I=0 & I \\ \end{array} (;) & (n) \\ I \\ \end{array} (;) \\ K_{n}(;) & d \\ \end{array} = K_{n}(;):$$

Then the marginals  $p_{I}^{(n)}$  of  $p_{n}$  are given by the determinant formulas  $Z_{I}^{(n)}(1, \dots, 1) := p_{n}(1, \dots, n)d_{I+1} \dots d_{n}$   $= (n \dots (n + 1))^{-1} \det fK_{n}(j, k)g_{j,k=1}^{I}$ 

Determinant formulas imply:

(a) 
$$\mathbf{E}fn^{-1}N_{n}[']g = {\overset{\mathsf{R}}{\overset{\circ}{}}}'()_{n}()d; _{n}() = K_{n}(;);$$

(b) 
$$\operatorname{Var} fN_n[']g = \frac{1}{2} R^R ('(1) '(2))^2 K_n^2(1; 2) d_1 d_2;$$

(b) **Var**
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$$\operatorname{Var} fN_{n}[']g = \frac{1}{2} \overset{R R}{('(1) '(2))^{2}} K_{n}^{2}(1; 2) d_{1} d_{2};$$
  
(c)  $\operatorname{P} fN_{n}() = 0g = \det(1 K_{n}()), \text{ where}$   
 $Z$   
 $(K_{n}()f)() = K_{n}(;)f()$ 

On the other hand, in *P., Shcherbina 97, 07* the universality of the local bulk regime of hermitian matrix models is proved for globally  $C^2$  and locally  $C^3$  potentials (see above theorem), basing on the orthogonal polynomial techniques, in particular on the above integral representation for  $K_n$ , but NOT using asymptotics of corresponding orthogonal polynomials.

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In *P.*, Shcherbina, 97 sin(x) = (x) is obtained via its Taylor expansion.

In *P., Shcherbina*, 07 Sin(x) =(x) is obtained as solution of a non-linear integro-differential equation.

# 2.2 Integro-differential Equation for Rescaled Reproducing Kernel

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Differentiate the representation with respect to X to obtain the identity

$$\frac{\overset{@}{=}}{\overset{@}{=}} K_n(x;y) = \frac{1}{2} V^{\theta}(_0 + x=n) K_n(x;y) + \frac{K_n(x^{\theta};x^{\theta}) K_n(x;y) - K_n(x;x^{\theta}) K_n(x^{\theta};y)}{x - x^{\theta}} dx^{\theta}$$

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We prove next that under the conditions of theorem we have uniformly in  $jxj; jyj < L; \quad 0 \ 2 \ [a + d; b \ d]:$ 

$$\frac{@}{@x}K_n(x;y) + \frac{@}{@y}K_n(x;y) \qquad C \quad n^{1=8} + jx \quad yjn^2 ;$$

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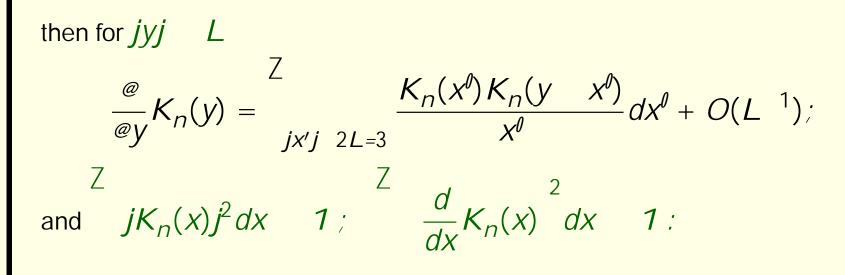
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Now, if

$$K_{n}(x) = K_{n}(x; 0)\mathbf{1}_{jxj} + K_{n}(L; 0)(1 + L - x)\mathbf{1}_{L < x - L + 1} + K_{n}(L; 0)(1 + L + x)\mathbf{1}_{L - 1 - x < - L};$$



#### 2.3 Asymptotic Solution of Equation

Consider the Fourier transform

Then we have from  $n^{-1}K_n(;) =$ 

Since  $K_n$  is "asymptotically even" Z  $j \aleph_n(p) \qquad \aleph_n(p) j^2 dp$  Z $= 2 \qquad j K_n(x) \qquad K_n(-x) j^2 dx \qquad Cn^{-1-8} \log^3 n$ 

we obtain the Fourier form of the above integro-differential equation:

$$Z = Z \xrightarrow{p}_{0} \aleph_{n}(p^{0}) dp^{0} \quad p \in {}^{ipy}dp = O(L^{-1}); jyj \quad L=3:$$

Since  $K_n$  is "asymptotically even"  $j \aleph_n(p) \quad \aleph_n(p) j^2 dp$  $= 2 \quad iK_n(x) \quad K_n(x)i^2 dx \quad Cn^{-1=8} \log^3 n$ we obtain the Fourier form of the above integro-differential equation:  $\aleph_n(p) \stackrel{Z \ p}{\longrightarrow} \aleph_n(p^{\theta}) dp^{\theta} \quad p \ e^{ipy} dp = O(L^{-1}); jyj \quad L=3:$ Besides, since  $K_n$  is positive definite  $\aleph_n$  is "asymptotically non-negative": 7  $\aleph_n(p)j\hat{f}(p)j^2dp$   $Cjjfjj_{L^2(\mathbb{R})}^2(n^{1=8}\log^4 n + O(L^{-1})):$ 

$$F_n(p) = \int_0^{Z_p} \aleph_n(p^{\ell}) dp^{\ell}$$

Since  $p \not{R}_n 2 L^2(\mathbb{R})$ , the sequence  $fF_n g$  consists of functions that are of uniformly bounded variation, uniformly bounded and equicontinuous on  $\mathbb{R}$ . Thus  $fF_n g$ 

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(a) F is bounded and continuous;

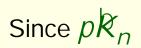
(b) F(p) = F(p);

$$F_n(p) = \int_0^{\perp} p \aleph_n(p^{\ell}) dp^{\ell}$$

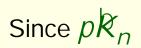
Since  $p \aleph_n 2 L^2(\mathbb{R})$ , the sequence  $f F_n g$  consists of functions that are of uniformly bounded variation, uniformly bounded and equicontinuous on R. Thus  $fF_{n,q}$  is a compact family with respect to the uniform convergence. Hence, the limit F of any subsequence  $fF_{D_k}g$  possesses the properties:

- (a) F is bounded and continuous;
- (b) F(p) = F(p);(c)  $F(p) = F(p^{\theta}), \text{ if } p = p^{\theta};$

$$F_n(p) = \int_0^{\mathbb{Z}_p} \aleph_n(p^{\ell}) dp^{\ell}$$



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(e) F satisfies the following equation, valid for any smooth function g of compact support:
Z

$$(F(p) \ p)g(p)dF(p) = 0:$$

The last property implies that F(p) = p or F(p) = const, hence it follows from (a) – (c) that

$$F(p) = p \mathbf{1}_{jpj \ p_0} + \text{sign}(p) \mathbf{1}_{jpj \ p_0}$$

where  $p_0 = (0)$  by (d).

We conclude that the equation is uniquely soluble, thus the sequence  $fF_ng$  converges uniformly on any compact to the above F. This imply the weak

convergence of the sequence  $fK_ng$  to the function

$$\mathcal{K}(x) = \frac{\sin((x))}{(x)}$$

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$$K(x) = \frac{\sin((x))}{(x)}$$

But weak convergence implies

$$\lim_{n! \to 1} K_n(x; y) = K (x + y);$$

uniformly in (x; y), varying on a compact set of  $\mathbb{R}^2$ , because  $\frac{d}{dx}K_n 2 L^2(\mathbb{R})$ :